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Zero-product preserving additive maps on symmetric operator spaces and self-adjoint operator spaces[☆]

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Abstract

In this note, we characterize the additive maps on the space of symmetric operators and the space of self-adjoint operators which preserve zero-products in both directions, and the additive maps on the space of self-adjoint operators which preserve Jordan zero-products in both directions. We also give a complete classification of additive maps on the von Neumann algebra of all bounded linear operators acting on a Hilbert space which preserve square zero in both directions or preserve operators annihilated by a polynomial in both directions.

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1. Introduction and statement of main results

Linear preserver problems concern characterization of linear maps on matrix algebras that leave certain properties, functions, subsets or relations invariant. This subject has attracted the attention of many mathematicians during last century (see the survey paper [6]). In the last decades interest in similar questions on operator algebras or operator spaces over infinite-dimensional Banach spaces has also been growing. Compared with the linear preserver problems, a more general task would be to consider an algebra as only a ring, and to assume the maps being additive only. It is surprising that, in many cases, the gap between linear maps and additive maps is a very big one, and it is much more difficult to deal with the additive ones. Moreover, it has been found from many recent results that it is additive preservers that play a more important role in the study of general preserver problems.

The study of zero-product preserving linear or additive maps between operator algebras is a topic that interests many authors greatly. A map Φ between two subsets \mathcal{A} , \mathcal{B} of some algebras is called a zero-product preserver (in both directions) if $\Phi(T)\Phi(S) = 0$ whenever (if and only if) $TS = 0$ for all $T, S \in \mathcal{A}$. Let $\mathcal{C}(X)$ be the algebra of all continuous complex functions defined on a compact Hausdorff space X . Then a linear zero-product preserver $\Phi : \mathcal{C}(X) \rightarrow \mathcal{C}(Y)$ is of the form $\Phi(f) = hf \circ \sigma$, where $h \in \mathcal{C}(Y)$ can be zero somewhere and $\sigma : Y \rightarrow X$ is a general continuous map, see [8]. In [5], the authors studied the continuous linear zero-product preserving maps in a more general algebraic settings, especially in the C^* -algebras. It was shown in [3] that every bijective linear map between two unital standard operator algebras which preserves zero-products in both directions is automatically continuous and is a scalar multiple of an algebra isomorphism. Linear or additive maps preserving zero-products on nest algebras were discussed in [7]. However, there are few papers discussing the zero-product preserving maps between operator spaces. Motivated by this, we study in this paper the additive maps on the symmetric operator space and the self-adjoint operator space which preserve zero-products in both directions.

We say that Φ is a Jordan zero-product preserving map if $\Phi(T)\Phi(S) + \Phi(S)\Phi(T) = 0$ whenever $TS + ST = 0$. We know that many operator spaces bear a Jordan ring structure. So it is also interesting to classify the additive maps that preserve Jordan zero-products. However, it seems that there was no paper appeared on this topic.

In the present note, we characterize the additive maps on the space of symmetric operators and the space of self-adjoint operators which preserve zero-products in both directions (Theorems 1.1 and 1.2) by using the results in [2,4] concerning the characterizations of maps preserving adjacency. We also give a complete classification of additive maps on the von Neumann algebra of all bounded linear operators acting on an infinite-dimensional Hilbert space which preserve square-zeros in both directions or preserve operators annihilated by a polynomial in both directions (Lemma 2.3 and Corollary 2.4), improving some results in [1]. This enables us to get

a characterization of the additive maps on the space of self-adjoint operators which preserve Jordan zero-products in both directions (Theorem 1.3).

Before stating our main results, we first introduce some notations. Let H be an infinite-dimensional Hilbert space over the complex field \mathbb{C} . Denote $\mathcal{B}(H)$ the algebra of all bounded linear operators on H . Let T^* denote the conjugate of T for every $T \in \mathcal{B}(H)$ and $\mathcal{S}^a(H) = \{T \in \mathcal{B}(H) | T = T^*\}$ the real linear space of all self-adjoint operators. Fix an orthonormal basis $\{e_\lambda | \lambda \in A\}$ of H . For any $x \in H$, we have $x = \sum_{\lambda \in A} \langle x, e_\lambda \rangle e_\lambda$ and define $\bar{x} = \sum_{\lambda \in A} \langle e_\lambda, x \rangle e_\lambda$. For every pair $T, S \in \mathcal{B}(H)$, if $\langle Te_\lambda, e_\mu \rangle = \langle Se_\mu, e_\lambda \rangle$ holds for all $\lambda, \mu \in A$, then we say that S is the transpose of T and denotes $S = T^t$. (The same notion can be defined easily for operators between two different Hilbert spaces.) Let J be the conjugate linear operator on H defined by $Jx = \bar{x}$ for every $x \in H$. It is clear that $T^t = JT^*J$. T is said to be symmetric (relative to the basis $\{e_\lambda | \lambda \in A\}$ of H) if $T = T^t$. We denote by $\mathcal{S}^y(H)$ the linear subspace of all symmetric operators in $\mathcal{B}(H)$ relative to some given basis $\{e_\lambda | \lambda \in A\}$. It is clear that every symmetric rank-one operator has the form $x \otimes \bar{x}$ for some $x \in H$ and vice versa. In [2, Lemma 2.1], it was shown that every finite-rank symmetric operator can be expressed as a sum of finitely many rank-one symmetric operators. Recall that an additive map A on H is said to be τ -linear if τ is an automorphism of \mathbb{C} and $A(\alpha x) = \tau(\alpha)Ax$ holds for all $\alpha \in \mathbb{C}$ and $x \in H$. If τ is the conjugation, that is, if $\tau(\alpha) \equiv \bar{\alpha}$, the τ -linear maps are said to be conjugate linear. For a conjugate linear operator $A : H \rightarrow H$, its transpose A^t can be similarly defined and is a conjugate linear operator on H .

Now let us state our main results.

Theorem 1.1. *Let H and K be infinite-dimensional complex Hilbert spaces. Let $\Phi : \mathcal{S}^y(H) \rightarrow \mathcal{S}^y(K)$ be an additive surjection. Then Φ preserves zero-products in both directions if and only if there exist a nonzero scalar c and a linear or conjugate linear invertible operator $A : H \rightarrow K$ satisfying $AA^t = I$ such that*

$$\Phi(T) = cATA^t$$

for all $T \in \mathcal{S}^y(H)$.

Theorem 1.2. *Let H and K be infinite-dimensional complex Hilbert spaces. Let $\Phi : \mathcal{S}^a(H) \rightarrow \mathcal{S}^a(K)$ be an additive surjection. Then Φ preserves zero-products in both directions if and only if there exist a nonzero real scalar c and a unitary or conjugate unitary operator $U : H \rightarrow K$ such that*

$$\Phi(T) = cUTU^*$$

for all $T \in \mathcal{S}^a(H)$.

Theorem 1.3. *Let H and K be infinite-dimensional complex Hilbert spaces. Let $\Phi : \mathcal{S}^a(H) \rightarrow \mathcal{S}^a(K)$ be an additive surjection. Then Φ preserves Jordan*

zero-products in both directions if and only if there exist a nonzero real scalar c and a unitary or conjugate unitary operator $U : H \rightarrow K$ such that

$$\Phi(T) = cUTU^*$$

for all $T \in \mathcal{S}^a(H)$.

We remark that it seems not obvious that the condition that Φ preserves zero-products in both directions is equivalent to the condition that Φ preserves Jordan zero-products in both directions for an additive surjection $\Phi : \mathcal{S}^a(H) \rightarrow \mathcal{S}^a(K)$. However, this fact is an immediate consequence of Theorems 1.2 and 1.3.

2. Proofs of the results

In this section, we devote to prove our main results stated in previous section, and also give some relative results concerning the matrix algebras, square-zero additive preservers as well as polynomial-zero additive preservers.

For the sake of convenience, we assume that $K = H$ in the proofs. Now let us introduce some more notations. For $S \in \mathcal{S}^y(H) \setminus \{0\}$ (or $S \in \mathcal{S}^a(H) \setminus \{0\}$), denote $\mathcal{R}(S)$ and $\overline{\mathcal{R}(S)}$ the range of S and the closure of the range of S , respectively. $\mathcal{R}(S)^\perp$ stands for the orthogonal complement of $\mathcal{R}(S)$. Put

$$S^\top = \{T \in \mathcal{S}^y(H) \setminus \{0\} | TS = 0\} \quad (\text{or } S^\top = \{T \in \mathcal{S}^a(H) \setminus \{0\} | TS = 0\}).$$

Proof of Theorem 1.1. It is obvious that the “if” part of the theorem is true. We need only to check the “only if” part.

Assume that Φ is an additive surjection and preserves zero-products in both directions. It is easy to verify that Φ is injective. So, in the following, we always assume that Φ is a bijection. We proceed in steps.

Step 1. For $S \in \mathcal{S}^y(H)$, $S^\top = \emptyset$ if and only if $\overline{\mathcal{R}(S)} = H$. And S^\top consists of rank-one symmetric operators if and only if $\mathcal{R}(S)^\perp$ is one dimensional.

If $\overline{\mathcal{R}(S)} = H$, then for any $T \in S^\top$, $TS = 0 \Rightarrow T = 0$. So $S^\top = \emptyset$. On the other hand, assume $S^\top = \emptyset$. If $\overline{\mathcal{R}(S)} \neq H$, then there exists a nonzero vector $x \in \mathcal{R}(S)^\perp$ such that $\bar{x} \otimes xS = 0$, leading to a contradiction.

If S^\top consists of rank-one symmetric operators but $\mathcal{R}(S)^\perp$ is not one dimensional, by the fact just proved, we must have $\dim(\mathcal{R}(S)^\perp) \geq 2$. So we can choose linearly independent elements $x_1, x_2 \in \mathcal{R}(S)^\perp$ such that $(\bar{x}_1 \otimes x_1 + \bar{x}_2 \otimes x_2)S = 0$, a contradiction. Conversely, assume that $\mathcal{R}(S)^\perp$ is one dimensional. For any $T \in S^\top$, $TS = 0$ implies $\overline{\mathcal{R}(S)} \subseteq \ker(T)$. Then it follows from $\mathcal{R}(JTJ) = \overline{\mathcal{R}(T^*)} = (\ker(T))^\perp \subseteq \mathcal{R}(S)^\perp$ that JTJ is of rank-one. Hence T is of rank-one.

Step 2. For any $x \in H \setminus \{0\}$, $\{\lambda x \otimes \bar{x} | \lambda \in \mathbb{C} \setminus \{0\}\} = S^\top$ for some $S \in \mathcal{S}^y(H)$.

If $\langle x, \bar{x} \rangle \neq 0$, without loss of generality, assume $\langle x, \bar{x} \rangle = 1$. Let $S = I - x \otimes \bar{x}$, where I is the identity in $\mathcal{B}(H)$. Obviously $x \otimes \bar{x} \in S^\top$ and $Sx = 0$. For any $z \in \ker(S)$, from $Sz = z - \langle z, \bar{x} \rangle x = 0$, one gets $\ker(S) \subseteq [x]$. Hence $\ker(S) = [x]$ and $\ker(S)$ is one dimensional. Since $S = JS^*J$ and $\mathcal{R}(S)^\perp = \ker(S^*)$, $\dim(\mathcal{R}(S)^\perp) = \dim(\ker(S^*)) = \dim(\ker(S)) = 1$. By Step 1, S^\top consists of rank-one symmetric operators. For any $u \otimes \bar{u}$, it follows from $u \otimes \bar{u}S = 0$ that $u \in \ker(S) = [x]$. Consequently, u and x are linearly dependent, and $S^\top = \{\lambda x \otimes \bar{x} | \lambda \in \mathbb{C} \setminus \{0\}\}$.

If $\langle x, \bar{x} \rangle = 0$, then $x + \bar{x} \neq 0$. Assume $\langle x, x \rangle = 1$ and let $S = I + \bar{x} \otimes x - (x + \bar{x}) \otimes (x + \bar{x})$. Then $x \otimes \bar{x} \in S^\top$ and $x \in \ker(S)$. For any $z \in \ker(S)$, it follows from $Sz = z + \langle z, x \rangle \bar{x} - \langle z, x + \bar{x} \rangle (x + \bar{x}) = 0$ that $z = \langle z, \bar{x} \rangle \bar{x} + \langle z, x + \bar{x} \rangle x$. Since $S\bar{x} = -x$, we have $Sz = \langle z, \bar{x} \rangle S\bar{x} + \langle z, x + \bar{x} \rangle Sx = -\langle z, \bar{x} \rangle x = 0$. It follows that $\langle z, \bar{x} \rangle = 0$ and $z = \langle z, x \rangle x$. So $\ker(S) = [x]$. As in the case that $\langle x, \bar{x} \rangle \neq 0$, we have $S^\top = \{\lambda x \otimes \bar{x} | \lambda \in \mathbb{C} \setminus \{0\}\}$.

Step 3. If $S^\top \neq \emptyset$, then $\mathcal{R}(S)^\perp$ is one dimensional if and only if there is no $P \in \mathcal{S}^y(H)$ such that $\emptyset \neq P^\top \subset S^\top$.

If $\mathcal{R}(S)^\perp$ is one dimensional but there exists $P \in \mathcal{S}^y(H)$ such that $\emptyset \neq P^\top \subset S^\top$, then for any $T \in P^\top$, it follows from $TP = 0$ and $TS = 0$ that both $\mathcal{R}(P)$ and $\mathcal{R}(S)$ are subset of $\ker(T)$. Since $\mathcal{R}(S)^\perp$ is one dimensional and $T \neq 0$, one gets $\ker(T) = \overline{\mathcal{R}(S)}$. So $\mathcal{R}(P) \subseteq \overline{\mathcal{R}(S)}$ and for any $T \in S^\top$, $TS = 0$ implies $TP = 0$, a contradiction.

Now assume that there is no $P \in \mathcal{S}^y(H)$ such that $\emptyset \neq P^\top \subset S^\top$ but $\mathcal{R}(S)^\perp$ is not one dimensional. There are two cases to consider.

Case 1. There exists $x \in \mathcal{R}(S)^\perp$ such that $\langle x, \bar{x} \rangle \neq 0$. Without loss of generality, we may assume $\langle x, \bar{x} \rangle = 1$. Let $P = I - \bar{x} \otimes x$; then $\bar{x} \otimes x \in P^\top$ and $P^\top \neq \emptyset$. For any $y \in \mathcal{R}(S)$, we have $Py = y$. Hence $\mathcal{R}(S) \subseteq \mathcal{R}(P)$ and $P^\top \subseteq S^\top$. Since $\dim \mathcal{R}(S)^\perp \geq 2$, there exists $x_0 \in \mathcal{R}(S)^\perp$ such that x_0 and x are linearly independent. Obviously, $\bar{x}_0 \otimes x_0 \in S^\top$, but $\bar{x}_0 \otimes x_0 P = \bar{x}_0 \otimes (x_0 - \langle \bar{x}, x_0 \rangle x) \neq 0$, which contradicts to the assumption that there is no $P \in \mathcal{S}^y(H)$ such that $\emptyset \neq P^\top \subset S^\top$.

Case 2. $\langle x, \bar{x} \rangle = 0$ for all $x \in \mathcal{R}(S)^\perp$. For any unit vectors $x, x_1 \in \mathcal{R}(S)^\perp$, it follows from $\langle x + x_1, \bar{x} + \bar{x}_1 \rangle = 0$ that $\langle x, \bar{x}_1 \rangle = 0$. This means that $\bar{x}_1 \in \overline{\mathcal{R}(S)}$ holds for every $x_1 \in \mathcal{R}(S)^\perp$. Let $P = I + x_1 \otimes \bar{x}_1 - (x_1 + \bar{x}_1) \otimes (x_1 + \bar{x}_1)$. Since $\bar{x}_1 \otimes x_1 P = 0$, $P^\top \neq \emptyset$. It follows from $Pz = z$ for any $z \in \mathcal{R}(S) \ominus [\bar{x}_1]$ and $P(-x_1) = \bar{x}_1$ that $\mathcal{R}(S) \subseteq \mathcal{R}(P)$ and hence $P^\top \subseteq S^\top$. Taking $x_2 \in \mathcal{R}(S)^\perp$ such that x_2, x_1 are linearly independent, it is easy to verify that $\bar{x}_2 \otimes x_2 \in S^\top$ but $\bar{x}_2 \otimes x_2 P \neq 0$, again a contradiction.

Step 4. Φ preserves rank-one symmetric operators in both directions. Furthermore there exist an automorphism τ of \mathbb{C} , a bijective τ -linear operator A on H such that $\Phi(x \otimes \bar{x}) = Ax \otimes \overline{Ax}$ holds for every rank-one symmetric operator $x \otimes \bar{x}$.

For any rank-one symmetric operator $x \otimes \bar{x}$, $x \in H \setminus \{0\}$, by Step 2, there exists $S \in \mathcal{S}^y(H)$ such that $\{\alpha x \otimes \bar{x} | \alpha \in \mathbb{C} \setminus \{0\}\} = S^\top$. By Steps 2, 3 and the properties of Φ , one gets that $\Phi(\{\alpha x \otimes \bar{x} | \alpha \in \mathbb{C} \setminus \{0\}\}) = \Phi(S^\top) = \Phi(S)^\top$ which consists

of rank-one symmetric operators. Thus Φ preserves rank-one symmetric operators in both directions, and therefore, preserves the adjacency in both directions by the additivity. (Recall that T and S are adjacent if $T - S$ is of rank-one.) Then the last assertion easily follows from [2, Theorem 1], which states that a surjective map $\Psi : \mathcal{S}_F^y(H) \rightarrow \mathcal{S}_F^y(H)$ preserves adjacency in both directions if and only if there exists an automorphism τ of \mathbb{C} , a τ -linear operator A on H and an operator $X_0 \in \mathcal{S}_F^y(H)$ such that $X \mapsto \Psi(X) - X_0$ is additive and

$$\Psi(x \otimes \bar{x}) = Ax \otimes \overline{Ax} + X_0$$

for all $x \in H$.

Step 5. There exists $c \in \mathbb{C} \setminus \{0\}$ such that $\langle Ax, \overline{Ay} \rangle = c\tau(\langle x, \bar{y} \rangle)$ holds for all $x, y \in H$.

If we can prove that $\langle Ax, \overline{Ax} \rangle = c\tau(\langle x, \bar{x} \rangle)$, then it follows from $\langle Ax + Ay, \overline{Ax} + \overline{Ay} \rangle = c\tau(\langle x + y, \bar{x} + \bar{y} \rangle)$ and $\langle x, \bar{y} \rangle = \langle y, \bar{x} \rangle$ that $\langle Ax, \overline{Ay} \rangle = c\tau(\langle x, \bar{y} \rangle)$. So we only prove that $\langle Ax, \overline{Ax} \rangle = c\tau(\langle x, \bar{x} \rangle)$ holds for all $x \in H$.

If $\langle x, \bar{y} \rangle = 0$, then $(y \otimes \bar{y})(x \otimes \bar{x}) = 0$, this implies that $(Ay \otimes \overline{Ay})(Ax \otimes \overline{Ax}) = 0$ and $\langle Ax, \overline{Ay} \rangle = 0$. Especially, when $y = \bar{x}$, we have $\langle x, \bar{x} \rangle = 0$ implies $\langle Ax, \overline{Ax} \rangle = 0$. Fix an $x_0 \in H$ such that $\langle x_0, \bar{x}_0 \rangle = 1$ and let $c = \langle Ax_0, \overline{Ax_0} \rangle$. Obviously $c \neq 0$. For any $x \in H$ with $\langle x, \bar{x} \rangle = 1$, if $\langle x, \bar{x}_0 \rangle = 0$, then it follows from $\langle ix + x_0, \overline{ix + x_0} \rangle = 0$ that $\langle A(ix + x_0), \overline{A(ix + x_0)} \rangle = 0$ and $\langle Aix, \overline{Aix} \rangle = -\langle Ax_0, \overline{Ax_0} \rangle$, where “ i ” is the imaginary unit. Since $\tau(i)^2 = -1$, we have $\langle Ax, \overline{Ax} \rangle = \langle Ax_0, \overline{Ax_0} \rangle = c\tau(\langle x, \bar{x} \rangle)$. If $\langle x, \bar{x}_0 \rangle \neq 0$, we can always find $y \in [x, x_0, \bar{x}, \bar{x}_0]^\perp$ with $\langle y, \bar{y} \rangle = 1$ (otherwise, we would have a contradiction that $\langle y_1, \bar{y}_2 \rangle = 0$ for all $y_1, y_2 \in [x, x_0, \bar{x}, \bar{x}_0]^\perp$). It follows from $\langle y, \bar{x} \rangle = 0$ that $\langle Ax, \overline{Ax} \rangle = \langle Ay, \overline{Ay} \rangle$ and from $\langle y, \bar{x}_0 \rangle = 0$ that $\langle Ay, \overline{Ay} \rangle = \langle Ax_0, \overline{Ax_0} \rangle$. Consequently, $\langle Ax, \overline{Ax} \rangle = \langle Ax_0, \overline{Ax_0} \rangle = c\tau(\langle x, \bar{x} \rangle)$. Hence, for any $x \in H$, we have $\langle Ax, \overline{Ax} \rangle = c\tau(\langle x, \bar{x} \rangle)$.

In the sequel, without loss of the generality, we assume that $c = 1$.

Step 6. τ is the identity or the conjugate map of \mathbb{C} . Consequently, A is a bounded linear or conjugate linear operator on H .

It is enough to prove that τ is continuous. Otherwise τ is unbounded in any neighborhood of 0. Choose a sequence $\{e_n\}_{n=1}^\infty$ in the fixed basis $\{e_\lambda | \lambda \in \Lambda\}$ of H . If $M = \sup_n \{\|Ae_n\|\} < \infty$, let $t_n \in \mathbb{C}$ be such that $|t_n| \leq 2^{-n}$ and $|\tau(t_n)| > n$. Let $x = \sum_{n=1}^\infty t_n e_n$; then $x \in H$, and

$$M\|Ax\| \geq |\langle Ax, \overline{Ae_n} \rangle| = |\tau(t_n)| > n$$

for all positive integer n , a contradiction. If $M = \sup_n \{Ae_n\} = \infty$, for each n , choose $r_n \in \mathbb{Q}$ (the field of rational numbers) with $\|Ae_n\| \leq r_n$ and $t_n \in \mathbb{C}$ with $|t_n/r_n| \leq 2^{-n}$ such that $|\tau(t_n/r_n^2)| > n$. Let $f_n = (1/r_n)e_n$ and $x = \sum_{n=1}^\infty (t_n/r_n)e_n \in H$. Then $\sup_n \|Af_n\| \leq 1$ but

$$\|Ax\| \geq |\langle Ax, \overline{Af_n} \rangle| = |\tau(\langle x, f_n \rangle)| = |\tau(t_n/r_n^2)| > n,$$

again a contradiction. So τ is continuous and hence A is linear or conjugate linear.

From the closed graph theorem, we know that A is bounded.

Step 7. $A^t = A^{-1}$ and $\Phi(T) = ATA^{-1}$ for every $T \in \mathcal{S}^y(H)$.

For every rank-one symmetric operator $x \otimes \bar{x}$ and for any $y \in H$,

$$\begin{aligned}\Phi(x \otimes \bar{x})y &= (Ax \otimes \overline{Ax})y = \langle y, \overline{Ax} \rangle Ax \\ &= \langle AA^{-1}y, \overline{Ax} \rangle Ax = \tau(\langle A^{-1}y, \bar{x} \rangle) Ax \\ &= A(\langle A^{-1}y, \bar{x} \rangle x) = A(x \otimes \bar{x})A^{-1}y.\end{aligned}$$

On the other hand, by Step 6, $\Phi(x \otimes \bar{x}) = Ax \otimes \overline{Ax} = A(x \otimes \bar{x})A^t$. So we have $A^t = A^{-1}$.

Let $\Psi(T) = A^{-1}\Phi(T)A$ for every $T \in \mathcal{S}^y(H)$. Obviously Ψ preserves zero-products in both directions and for any finite rank symmetric operator S , $\Psi(S) = S$. We have to prove that for every $T \in \mathcal{S}^y(H)$, $\Psi(T) = T$.

For any $x \in H \setminus \{0\}$, let $y = Tx$ and $a = \langle x, x \rangle$. If $\langle x, \bar{y} \rangle = t \neq 0$, let r be a square root of t and $S = \frac{y}{r} \otimes \overline{(\frac{y}{r})}$; if $\langle x, \bar{y} \rangle = 0$, let $S = \frac{y}{a} \otimes x + \bar{x} \otimes \overline{(\frac{y}{a})}$. Then it follows from $(T - S)x \otimes \bar{x} = 0$ that $\Psi(T)x = Sx = y = Tx$. So we have $\Psi(T) = T$ for every $T \in \mathcal{S}^y(H)$, completing the proof. \square

Checking the proof of Theorem 1.1 we see that the condition that H is infinite dimensional is only used in Step 6 to ensure that τ is continuous. Thus for finite dimensional case, one can easily checked that the following proposition is true. Denote $\mathcal{S}_n(\mathbb{C})$ the linear subspace of all symmetric matrices in $\mathcal{M}_n(\mathbb{C})$, the algebra of all $n \times n$ complex matrices.

Proposition 2.1. *Let $\phi : \mathcal{S}_n(\mathbb{C}) \rightarrow \mathcal{S}_n(\mathbb{C})$ ($n \geq 2$) be a surjective additive map. Then ϕ preserves zero-products in both directions if and only if there exist a nonzero scalar c , an automorphism τ of \mathbb{C} and an orthogonal matrix U such that $\phi(T) = cUT_\tau U^t$ for all $T \in \mathcal{S}_n(\mathbb{C})$. Here $T_\tau = (\tau(t_{ij}))_{n \times n}$ if $T = (t_{ij})_{n \times n}$.*

Proof of Theorem 1.2. We need only to check the “only if” part. Assume that $\Phi : \mathcal{S}^a(H) \rightarrow \mathcal{S}^a(H)$ is an additive surjection which preserves zero-products in both directions. It is easy to verify that Φ is injective. So Φ is bijective. Since the proof is similar to the proof of Theorem 1.1, we only give an outline and omit the details.

Step 1. For $S \in \mathcal{S}^a(H)$, $S^\top = \emptyset$ if and only if $\overline{\mathcal{R}(S)} = H$, and $\dim(\mathcal{R}(S)^\perp) = 1$ if and only if $S^\top = \{\alpha x \otimes x \mid \alpha \in \mathbb{R} \setminus \{0\}\}$ for some $x \in H$, $x \neq 0$, where \mathbb{R} is the field of real numbers.

Step 2. $S^\top \neq \emptyset$, then $\dim(\mathcal{R}(S)^\perp) = 1$ if and only if there is no $P \in \mathcal{S}^a(H)$ such that $\emptyset \neq P^\top \subset S^\top$.

The “only if” part is easy, we only check the “if” part and show that if $\dim(\mathcal{R}(S)^\perp) \neq 1$, then there exists $P \in \mathcal{S}^a(H)$ such that $\emptyset \neq P^\top \subset S^\top$. Assume $\dim(\mathcal{R}(S)^\perp) \neq 1$, by Step 1, $\dim(\mathcal{R}(S)^\perp) \geq 2$. Taking $x_1, x_2 \in \mathcal{R}(S)^\perp$ with

$\langle x_1, x_2 \rangle = 0$, let P be the projection with range $\overline{\mathcal{H}(S)} \oplus [x_1]$, then it is easy to verify that P is the desired operator.

Step 3. Φ preserves rank-one self-adjoint operators in both directions and there exist a nonzero real scalar a and bijective linear or conjugate linear operator A on H such that $\Phi(x \otimes x) = aAx \otimes Ax$ for every $x \in H$.

The assertion easily follows from Steps 1, 2, and [4, Theorem 1] which states that a surjective map $\Psi : \mathcal{S}_F^a(H) \rightarrow \mathcal{S}_F^a(H)$ preserves the adjacency in both directions if and only if there exist an $X_0 \in \mathcal{S}_F^a(H)$, a bijective linear or conjugate linear operator A on H and a scalar $c \in \mathbb{R} \setminus \{0\}$ such that $X \mapsto \Psi(X) - X_0$ is linear or conjugate-linear bijective map and

$$\Psi(x \otimes x) = cAx \otimes Ax + X_0$$

for all $x \in H$. Where $\mathcal{S}_F^a(H)$ stands for the set of all finite rank elements in $\mathcal{S}^a(H)$.

Step 4. A is bounded.

As that in Step 6 of the proof of Theorem 1.1, we claim that for any $x, y \in H$, $\langle Ax, Ay \rangle = b\langle x, y \rangle$ if A is linear or $\langle Ax, Ay \rangle = b\langle y, x \rangle$ if A is conjugate linear. From the closed graph theorem and the bijectivity of A , it follows that A is bounded. Furthermore we have $AA^* = bI$ and $b > 0$.

Let $U = \sqrt{\frac{1}{b}}A$ and $c = ab$. Then U is unitary or conjugate unitary and, for any $x \in H$, $\Phi(x \otimes x) = cUx \otimes xU^*$.

Step 5. For every $T \in \mathcal{S}^a(H)$, $\Phi(T) = cUTU^*$.

Let $\Psi(T) = \frac{1}{c}U^*\Phi(T)U$, then for any finite rank self-adjoint operator S , we have $\Psi(S) = S$. We have to prove that $\Psi(T) = T$ for every $T \in \mathcal{S}^a(H)$.

For any $x \in H \setminus \{0\}$, let P be the projection with range $[x, Tx]$ and $S = -PTP$, then $Sx = -Tx$. It follows from $(x \otimes x)(T + S) = 0$ that $\Psi(T)x = -Sx = Tx$, completing the proof. \square

Just similar to Theorem 1.1, the finite dimensional version of Theorem 1.2 is the following proposition. We denote $\mathcal{H}_n(\mathbb{C})$ the real linear subspace of all Hermitian matrices in $\mathcal{M}_n(\mathbb{C})$.

Proposition 2.2. Let $\phi : \mathcal{H}_n(\mathbb{C}) \rightarrow \mathcal{H}_n(\mathbb{C})$ ($n \geq 2$) be a surjective additive map. Then ϕ preserves zero-products in both directions if and only if there exist a non-zero real scalar c and a unitary matrix U such that either $\phi(T) = cUTU^*$ for all $T \in \mathcal{S}_n(\mathbb{C})$ or $\phi(T) = cU\bar{T}U^*$ for all $T \in \mathcal{S}_n(\mathbb{C})$. Where $\bar{T} = (\bar{t}_{ij})_{n \times n}$ if $T = (t_{ij})_{n \times n}$.

To prove Theorem 1.3, we need a lemma, in which, the equivalency of (2) and (3) there gives a characterization of additive maps on $\mathcal{B}(H)$ which preserve square-zero operators in both directions, improving a main result in [1, Theorem 3.1] by omitting the assumption that $\Phi(\mathbb{C}P) \subseteq \mathbb{C}\Phi(P)$ holds for every rank-1 idempotent. Recall that

Φ is said to be square-zero preserving (in both directions) if $\Phi(T)^2 = 0$ whenever (if and only if) $T^2 = 0$.

Lemma 2.3. *Let H and K be infinite-dimensional complex Hilbert spaces and let $\Phi : \mathcal{B}(H) \rightarrow \mathcal{B}(K)$ be a surjective additive map. Then the following statements are equivalent:*

- (1) Φ preserves Jordan zero-products in both directions.
- (2) Φ preserves square-zeroes in both directions.
- (3) *There exist a nonzero scalar c and a bounded invertible linear or conjugate linear operator $A : H \rightarrow K$ such that either $\Phi(T) = cATA^{-1}$ for all $T \in \mathcal{B}(H)$ or $\Phi(T) = cAT^*A^{-1}$ for all $T \in \mathcal{B}(H)$.*

Proof. Obviously an additive map on a ring preserving Jordan zero-products must be square-zero preserving. So we need only check (2) \Rightarrow (3). Assume that Φ preserves square-zeroes in both directions. Then, by what proved in [1], Φ is injective and $\Phi(I) = cI$ for some nonzero complex number c . Let $\Psi = c^{-1}\Phi$. Then, Ψ is idempotent preserving. Now a result of Kuzma [9, Corollary 3.3] ensures us the existence of the desired A such that $\Psi(T) = ATA^{-1}$ for all T or $\Psi(T) = AT^*A^{-1}$ for all T . \square

For a complex polynomial $p(t)$, a map Φ preserves operators annihilated by $p(t)$ (in both directions) if $p(\Phi(T)) = 0$ whenever (if and only if) $p(T) = 0$. Let $\mathcal{Z}(p)$ be the set of all roots of p and let $\mathcal{G}(p) = \{\lambda \in \mathbb{C} \setminus \{0\} \mid \lambda\mathcal{Z}(p) \subseteq \mathcal{Z}(p)\}$. The following corollary is immediate from Lemma 2.3 and the arguments in [1], which is an improvement of [1, Theorem 3.8].

Corollary 2.4. *Let H and K be infinite-dimensional complex Hilbert spaces and let $p(t)$ be a complex polynomial with degree greater than 1. Assume that $\Phi : \mathcal{B}(H) \rightarrow \mathcal{B}(K)$ is a surjective additive map. Then Φ preserves operators annihilated by $p(t)$ in both directions if and only if there exist a scalar $c \in \mathcal{G}(p)$ and a bounded invertible linear or conjugate linear operator $A : H \rightarrow K$ such that either $\Phi(T) = cATA^{-1}$ for all $T \in \mathcal{B}(H)$ or $\Phi(T) = cAT^*A^{-1}$ for all $T \in \mathcal{B}(H)$.*

Now we are ready to give the proof of Theorem 1.3.

Proof of Theorem 1.3. Define a map $\Psi : \mathcal{B}(H) \rightarrow \mathcal{B}(H)$ by

$$\Psi(A) = \Phi(S) + i\Phi(T)$$

for all $A \in \mathcal{B}(H)$ with $S = \frac{1}{2}(A + A^*)$, $T = \frac{1}{2i}(A - A^*) \in \mathcal{S}^a(H)$. Then it is easily verified that Ψ is a bijective additive map. Next, we show that Ψ preserves square-zeroes in both directions.

For any $A \in \mathcal{B}(H)$ with $A^2 = 0$, write $A = S + iT$, $S, T \in \mathcal{S}(H)$. Then it follows from $A^2 = 0$ that $S^2 - T^2 = 0$ and $ST + TS = 0$. Observe that $S^2 - T^2 =$

$\frac{1}{2}((S+T)(S-T) + (S-T)(S+T))$. Since Φ preserves Jordan zero-products, we have that $\Phi(S)^2 = \Phi(T)^2$ and $\Phi(S)\Phi(T) + \Phi(T)\Phi(S) = 0$. This implies that $\Psi(A)^2 = 0$. The same conclusion holds for Φ^{-1} and Ψ^{-1} . Consequently, Ψ preserves square-zero operators in both directions.

Applying Lemma 2.3, $\Psi(A) = cVA V^{-1}$ for all $A \in B(H)$ or $\Psi(A) = cVA^*V^{-1}$ holds for all $A \in B(H)$ for some nonzero scalar c and invertible linear or conjugate linear operator V on H . Since $\Psi|_{\mathcal{S}^a(H)} = \Phi$ and $\Phi(S) \in \mathcal{S}^a(H)$ for every $S \in \mathcal{S}^a(H)$, we see that $c \in \mathbb{R}$ and $V^*V = bI$ for some $b > 0$. Let $U = \sqrt{b^{-1}}V$; then $\Phi(S) = cUSU^*$ for all $S \in \mathcal{S}^a(H)$, completing the proof. \square

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